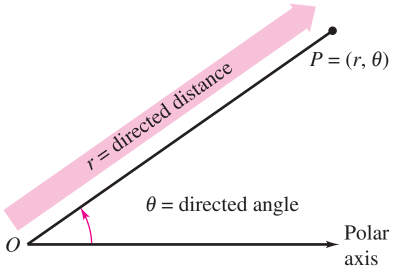
10.4 Polar Coordinates and Polar Graphs

- Understand the polar coordinate system.
- Rewrite rectangular coordinates and equations in polar form and vice versa.
- Sketch the graph of an equation given in polar form.
- Find the slope of a tangent line to a polar graph.
- Identify several types of special polar graphs.

Polar Coordinates

So far, you have been representing graphs as collections of points (x, y) on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form. In this section, you will study a coordinate system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in Figure 10.35. Then each point P in the plane can be assigned **polar coordinates** (r, θ), as follows.



Polar coordinates
Figure 10.35

$r =$ *directed distance* from O to P

$\theta =$ *directed angle*, counterclockwise from polar axis to segment $\overline{OP}$

Figure 10.36 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.

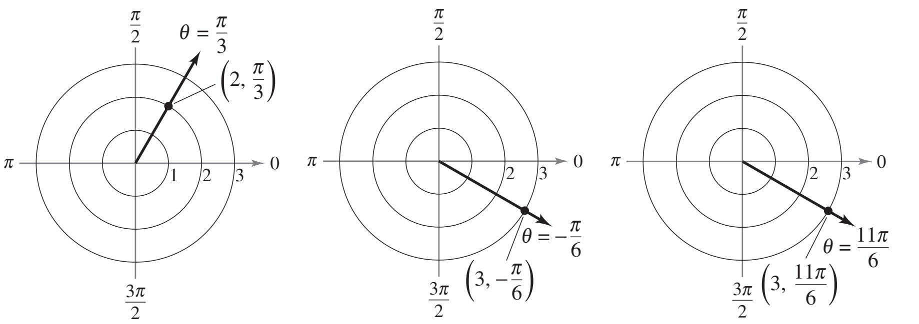


Figure 10.36

With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates. For instance, the coordinates

$$(r, \theta) \quad \text{and} \quad (r, 2\pi + \theta)$$

represent the same point [see parts (b) and (c) in Figure 10.36]. Also, because r is a *directed distance,* the coordinates

$$(r, \theta) \quad \text{and} \quad (-r, \theta + \pi)$$

represent the same point. In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi)$$

or

$$(r, \theta) = (-r, \theta + (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

POLAR COORDINATES

The mathematician credited with first using polar coordinates was James Bernoulli, who introduced them in 1691. However, there is some evidence that it may have been Isaac Newton who first used them.

Coordinate Conversion

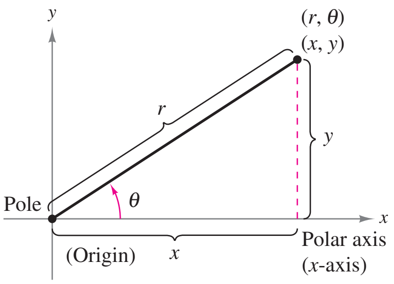
To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 10.37. Because (x, y) lies on a circle of radius r, it follows that

$$r^2 = x^2 + y^2.$$

Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan \theta = \frac{y}{x}, \quad \cos \theta = \frac{x}{r}, \quad \text{and} \quad \sin \theta = \frac{y}{r}.$$

You can show that the same relationships hold for $r < 0$.



Relating polar and rectangular coordinates
Figure 10.37

> **THEOREM 10.10 Coordinate Conversion**
>
> The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.
>
> | **Polar-to-Rectangular** | **Rectangular-to-Polar** |
> |---|---|
> | $x = r \cos \theta$ | $\tan \theta = \dfrac{y}{x}$ |
> | $y = r \sin \theta$ | $r^2 = x^2 + y^2$ |

EXAMPLE 1 Polar-to-Rectangular Conversion

a. For the point $(r, \theta) = (2, \pi)$,

$$x = r \cos \theta = 2 \cos \pi = -2 \quad \text{and} \quad y = r \sin \theta = 2 \sin \pi = 0.$$

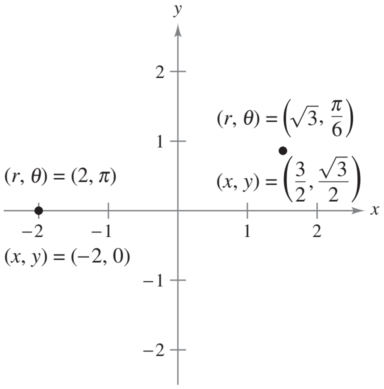
So, the rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = (\sqrt{3}, \pi/6)$,

$$x = \sqrt{3} \cos \frac{\pi}{6} = \frac{3}{2} \quad \text{and} \quad y = \sqrt{3} \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, the rectangular coordinates are $(x, y) = (3/2, \sqrt{3}/2)$.

See Figure 10.38.



To convert from polar to rectangular coordinates, let $x = r \cos \theta$ and $y = r \sin \theta$.
Figure 10.38

EXAMPLE 2 Rectangular-to-Polar Conversion

a. For the second-quadrant point $(x, y) = (-1, 1)$,

$$\tan \theta = \frac{y}{x} = -1 \quad \Longrightarrow \quad \theta = \frac{3\pi}{4}.$$

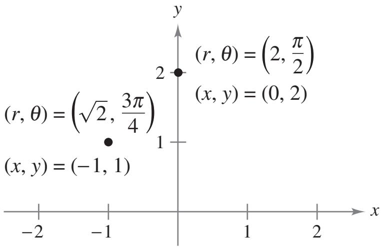
Because θ was chosen to be in the same quadrant as (x, y), you should use a positive value of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (1)^2} \\ &= \sqrt{2} \end{aligned}$$

This implies that *one* set of polar coordinates is $(r, \theta) = (\sqrt{2}, 3\pi/4)$.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose $\theta = \pi/2$ and $r = 2$, and one set of polar coordinates is $(r, \theta) = (2, \pi/2)$.

See Figure 10.39.



To convert from rectangular to polar coordinates, let $\tan \theta = y/x$ and $r = \sqrt{x^2 + y^2}$.
Figure 10.39

Polar Graphs

One way to sketch the graph of a polar equation is to convert to rectangular coordinates and then sketch the graph of the rectangular equation.

EXAMPLE 3 Graphing Polar Equations

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

a. $r = 2$ **b.** $\theta = \dfrac{\pi}{3}$ **c.** $r = \sec\theta$

Solution

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. So, this graph is a circle centered at the origin with a radius of 2. [See Figure 10.40(a).] You can confirm this by using the relationship $r^2 = x^2 + y^2$ to obtain the rectangular equation

$$x^2 + y^2 = 2^2. \qquad \text{Rectangular equation}$$

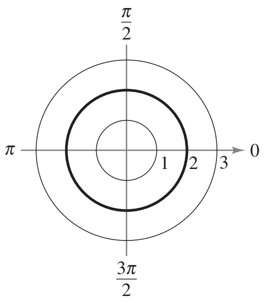
b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis. [See Figure 10.40(b).] You can confirm this by using the relationship $\tan\theta = y/x$ to obtain the rectangular equation

$$y = \sqrt{3}x. \qquad \text{Rectangular equation}$$

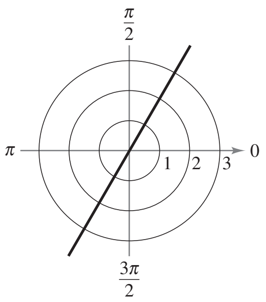
c. The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you can begin by converting to rectangular form using the relationship $r\cos\theta = x$.

$$r = \sec\theta \qquad \text{Polar equation}$$
$$r\cos\theta = 1$$
$$x = 1 \qquad \text{Rectangular equation}$$

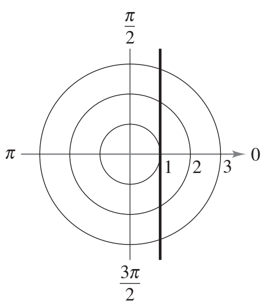
From the rectangular equation, you can see that the graph is a vertical line. [See Figure 10.40(c).]



(a) Circle: $r = 2$



(b) Radial line: $\theta = \dfrac{\pi}{3}$



(c) Vertical line: $r = \sec\theta$

Figure 10.40

TECHNOLOGY Sketching the graphs of complicated polar equations *by hand* can be tedious. With technology, however, the task is not difficult. If your graphing utility has a *polar* mode, use it to graph the equations in the exercise set. If your graphing utility doesn't have a *polar* mode, but does have a *parametric* mode, you can graph $r = f(\theta)$ by writing the equation as

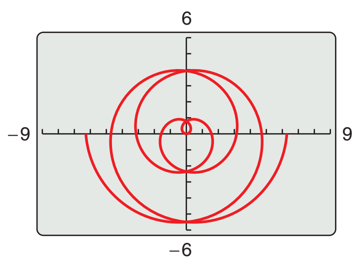
$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta.$$

For instance, the graph of $r = \frac{1}{2}\theta$ shown in Figure 10.41 was produced with a graphing calculator in parametric mode. This equation was graphed using the parametric equations

$$x = \frac{1}{2}\theta\cos\theta$$

$$y = \frac{1}{2}\theta\sin\theta$$

with the values of θ varying from -4π to 4π. This curve is of the form $r = a\theta$ and is called a **spiral of Archimedes.**



Spiral of Archimedes

Figure 10.41

REMARK One way to sketch the graph of $r = 2\cos 3\theta$ by hand is to make a table of values.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
r	2	0	-2	0	2

By extending the table and plotting the points, you will obtain the curve shown in Example 4.

EXAMPLE 4 Sketching a Polar Graph

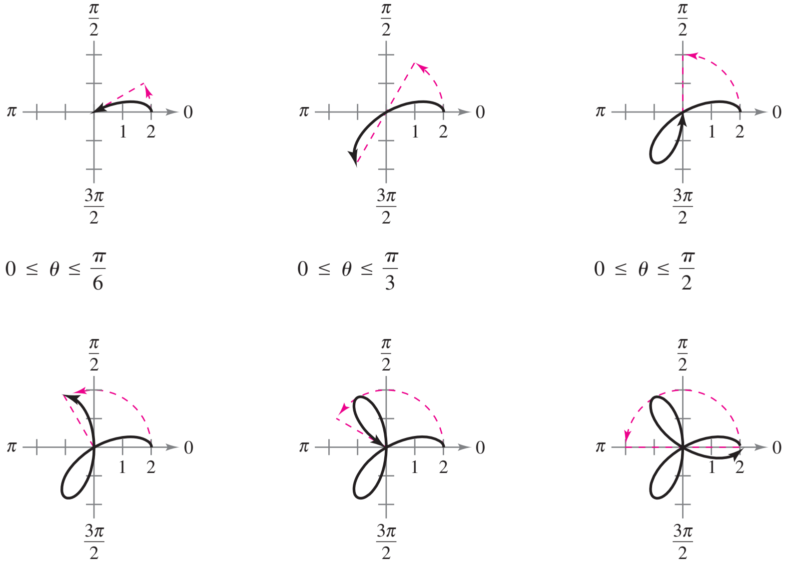
See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graph of $r = 2\cos 3\theta$.

Solution Begin by writing the polar equation in parametric form.

$$x = 2\cos 3\theta \cos\theta \quad \text{and} \quad y = 2\cos 3\theta \sin\theta$$

After some experimentation, you will find that the entire curve, which is called a **rose curve,** can be sketched by letting θ vary from 0 to π, as shown in Figure 10.42. If you try duplicating this graph with a graphing utility, you will find that by letting θ vary from 0 to 2π, you will actually trace the entire curve *twice.*



$0 \le \theta \le \frac{\pi}{6}$ $0 \le \theta \le \frac{\pi}{3}$ $0 \le \theta \le \frac{\pi}{2}$

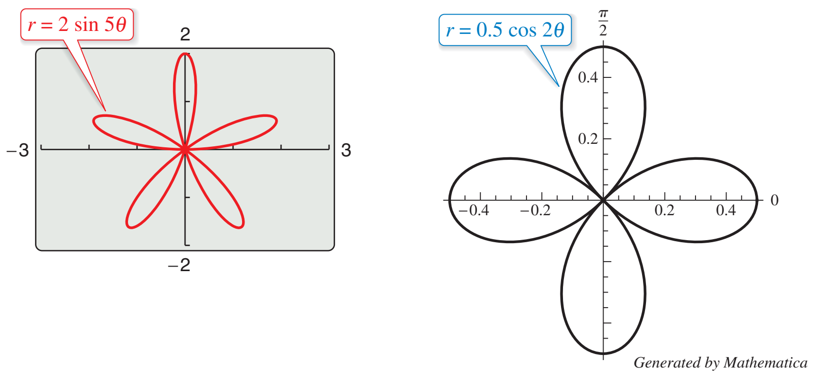
$0 \le \theta \le \frac{2\pi}{3}$ $0 \le \theta \le \frac{5\pi}{6}$ $0 \le \theta \le \pi$

Figure 10.42

Use a graphing utility to experiment with other rose curves. Note that rose curves are of the form

$$r = a\cos n\theta \quad \text{or} \quad r = a\sin n\theta.$$

For instance, Figure 10.43 shows the graphs of two other rose curves.



Rose curves
Figure 10.43

Slope and Tangent Lines

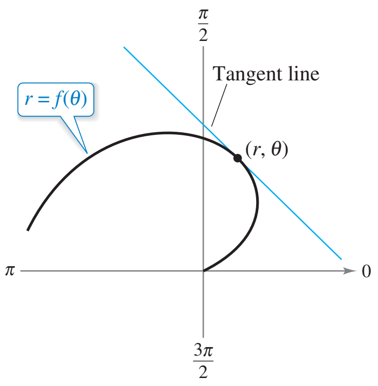
To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Using the parametric form of dy/dx given in Theorem 10.7, you have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

which establishes the next theorem.



Tangent line to polar curve
Figure 10.44

THEOREM 10.11 Slope in Polar Form

If f is a differentiable function of θ, then the *slope* of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ). (See Figure 10.44.)

From Theorem 10.11, you can make the following observations.

1. Solutions of $\dfrac{dy}{d\theta} = 0$ yield horizontal tangents, provided that $\dfrac{dx}{d\theta} \neq 0$.
2. Solutions of $\dfrac{dx}{d\theta} = 0$ yield vertical tangents, provided that $\dfrac{dy}{d\theta} \neq 0$.

If $dy/d\theta$ and $dx/d\theta$ are *simultaneously* 0, then no conclusion can be drawn about tangent lines.

EXAMPLE 5 Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$.

Solution Begin by writing the equation in parametric form.

$$x = r\cos\theta = \sin\theta\cos\theta$$

and

$$y = r\sin\theta = \sin\theta\sin\theta = \sin^2\theta$$

Next, differentiate x and y with respect to θ and set each derivative equal to 0.

$$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = \cos 2\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \quad \Rightarrow \quad \theta = 0, \frac{\pi}{2}$$

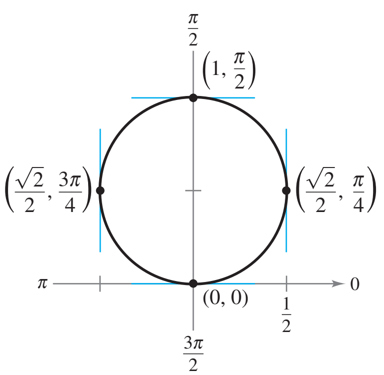
So, the graph has vertical tangent lines at

$$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) \quad \text{and} \quad \left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$$

and it has horizontal tangent lines at

$$(0, 0) \quad \text{and} \quad \left(1, \frac{\pi}{2}\right)$$

as shown in Figure 10.45.



Horizontal and vertical tangent lines of $r = \sin\theta$
Figure 10.45

EXAMPLE 6 Finding Horizontal and Vertical Tangent Lines

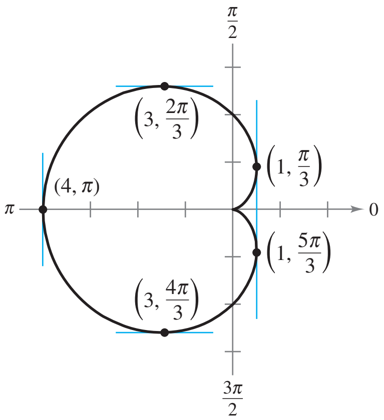
Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

Solution Let $y = r\sin\theta$ and then differentiate with respect to θ.

$$\begin{aligned} y &= r\sin\theta \\ &= 2(1-\cos\theta)\sin\theta \\ \frac{dy}{d\theta} &= 2[(1-\cos\theta)(\cos\theta) + \sin\theta(\sin\theta)] \\ &= 2(\cos\theta - \cos^2\theta + \sin^2\theta) \\ &= 2(\cos\theta - \cos^2\theta + 1 - \cos^2\theta) \\ &= -2(2\cos^2\theta - \cos\theta - 1) \\ &= -2(2\cos\theta + 1)(\cos\theta - 1) \end{aligned}$$

Setting $dy/d\theta$ equal to 0, you can see that $\cos\theta = -\frac{1}{2}$ and $\cos\theta = 1$. So, $dy/d\theta = 0$ when $\theta = 2\pi/3, 4\pi/3$, and 0. Similarly, using $x = r\cos\theta$, you have

$$\begin{aligned} x &= r\cos\theta \\ &= 2(1-\cos\theta)\cos\theta \\ &= 2\cos\theta - 2\cos^2\theta \\ \frac{dx}{d\theta} &= -2\sin\theta + 4\cos\theta\sin\theta \\ &= 2\sin\theta(2\cos\theta - 1). \end{aligned}$$



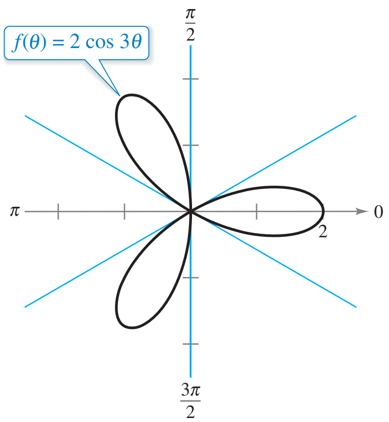
Horizontal and vertical tangent lines of $r = 2(1 - \cos\theta)$
Figure 10.46

Setting $dx/d\theta$ equal to 0, you can see that $\sin\theta = 0$ and $\cos\theta = \frac{1}{2}$. So, you can conclude that $dx/d\theta = 0$ when $\theta = 0, \pi, \pi/3$, and $5\pi/3$. From these results, and from the graph shown in Figure 10.46, you can conclude that the graph has horizontal tangents at $(3, 2\pi/3)$ and $(3, 4\pi/3)$, and has vertical tangents at $(1, \pi/3)$, $(1, 5\pi/3)$, and $(4, \pi)$. This graph is called a **cardioid.** Note that both derivatives ($dy/d\theta$ and $dx/d\theta$) are 0 when $\theta = 0$. Using this information alone, you don't know whether the graph has a horizontal or vertical tangent line at the pole. From Figure 10.46, however, you can see that the graph has a cusp at the pole.

Theorem 10.11 has an important consequence. If the graph of $r = f(\theta)$ passes through the pole when $\theta = \alpha$ and $f'(\alpha) \neq 0$, then the formula for dy/dx simplifies as follows.

$$\frac{dy}{dx} = \frac{f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha}{f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha} = \frac{f'(\alpha)\sin\alpha + 0}{f'(\alpha)\cos\alpha - 0} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

So, the line $\theta = \alpha$ is tangent to the graph at the pole, $(0, \alpha)$.



This rose curve has three tangent lines ($\theta = \pi/6$, $\theta = \pi/2$, and $\theta = 5\pi/6$) at the pole.
Figure 10.47

THEOREM 10.12 Tangent Lines at the Pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Theorem 10.12 is useful because it states that the zeros of $r = f(\theta)$ can be used to find the tangent lines at the pole. Note that because a polar curve can cross the pole more than once, it can have more than one tangent line at the pole. For example, the rose curve $f(\theta) = 2\cos 3\theta$ has three tangent lines at the pole, as shown in Figure 10.47. For this curve, $f(\theta) = 2\cos 3\theta$ is 0 when θ is $\pi/6$, $\pi/2$, and $5\pi/6$. Moreover, the derivative $f'(\theta) = -6\sin 3\theta$ is not 0 for these values of θ.

Special Polar Graphs

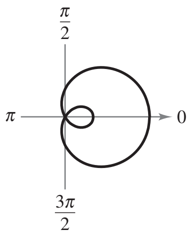
Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the polar equation of a circle having a radius of a and centered at the origin is simply $r = a$. Later in the text, you will come to appreciate this benefit. For now, several other types of graphs that have simpler equations in polar form are shown below. (Conics are considered in Section 10.6.)

Limaçons

$r = a \pm b \cos \theta$

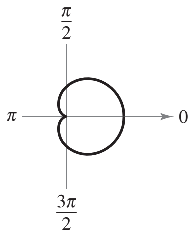
$r = a \pm b \sin \theta$

$(a > 0, b > 0)$



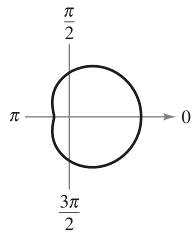
$\frac{a}{b} < 1$

Limaçon with inner loop



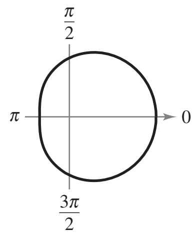
$\frac{a}{b} = 1$

Cardioid (heart-shaped)



$1 < \frac{a}{b} < 2$

Dimpled limaçon



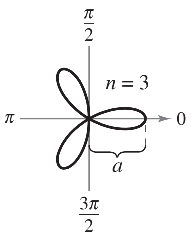
$\frac{a}{b} \geq 2$

Convex limaçon

Rose Curves

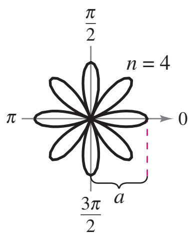
n petals when n is odd

$2n$ petals when n is even $(n \geq 2)$



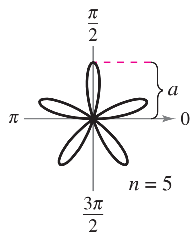
$r = a \cos n\theta$

Rose curve



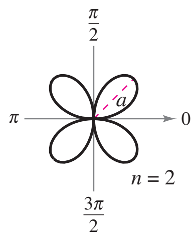
$r = a \cos n\theta$

Rose curve



$r = a \sin n\theta$

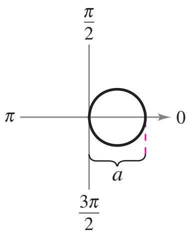
Rose curve



$r = a \sin n\theta$

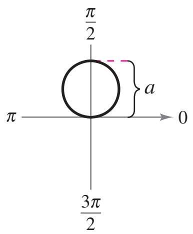
Rose curve

Circles and Lemniscates



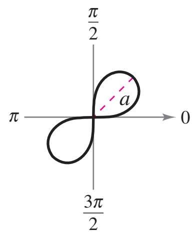
$r = a \cos \theta$

Circle



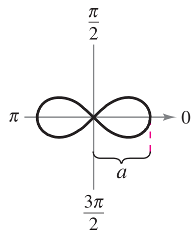
$r = a \sin \theta$

Circle



$r^2 = a^2 \sin 2\theta$

Lemniscate



$r^2 = a^2 \cos 2\theta$

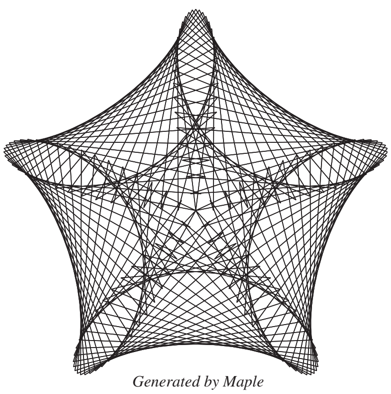
Lemniscate

TECHNOLOGY The rose curves described above are of the form $r = a \cos n\theta$ or $r = a \sin n\theta$, where n is a positive integer that is greater than or equal to 2. Use a graphing utility to graph

$$r = a \cos n\theta \quad \text{or} \quad r = a \sin n\theta$$

for some noninteger values of n. Are these graphs also rose curves? For example, try sketching the graph of

$$r = \cos \frac{2}{3}\theta, \quad 0 \leq \theta \leq 6\pi.$$



Generated by Maple

FOR FURTHER INFORMATION For more information on rose curves and related curves, see the article "A Rose is a Rose . . ." by Peter M. Maurer in *The American Mathematical Monthly.* The computer-generated graph at the left is the result of an algorithm that Maurer calls "The Rose." To view this article, go to *MathArticles.com.*

10.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Polar-to-Rectangular Conversion **In Exercises 1–10, plot the point in polar coordinates and find the corresponding rectangular coordinates for the point.**

1. $\left(8, \frac{\pi}{2}\right)$
2. $\left(-2, \frac{5\pi}{3}\right)$
3. $\left(-4, -\frac{3\pi}{4}\right)$
4. $\left(0, -\frac{7\pi}{6}\right)$
5. $\left(7, \frac{5\pi}{4}\right)$
6. $\left(-2, \frac{11\pi}{6}\right)$
7. $(\sqrt{2}, 2.36)$
8. $(-3, -1.57)$
9. $(-4.5, 3.5)$
10. $(9.25, 1.2)$

Rectangular-to-Polar Conversion **In Exercises 11–20, the rectangular coordinates of a point are given. Plot the point and find *two* sets of polar coordinates for the point for $0 \le \theta < 2\pi$.**

11. $(2, 2)$
12. $(0, -6)$
13. $(-3, 4)$
14. $(4, -2)$
15. $(-1, -\sqrt{3})$
16. $(3, -\sqrt{3})$
17. $(3, -2)$
18. $(3\sqrt{2}, 3\sqrt{2})$
19. $\left(\frac{7}{4}, \frac{5}{2}\right)$
20. $(0, -5)$

21. **Plotting a Point** Plot the point $(4, 3.5)$ when the point is given in
    (a) rectangular coordinates.
    (b) polar coordinates.

22. **Graphical Reasoning**
    (a) Set the window format of a graphing utility to rectangular coordinates and locate the cursor at any position off the axes. Move the cursor horizontally and vertically. Describe any changes in the displayed coordinates of the points.
    (b) Set the window format of a graphing utility to polar coordinates and locate the cursor at any position off the axes. Move the cursor horizontally and vertically. Describe any changes in the displayed coordinates of the points.
    (c) Why are the results in parts (a) and (b) different?

Rectangular-to-Polar Conversion **In Exercises 23–32, convert the rectangular equation to polar form and sketch its graph.**

23. $x^2 + y^2 = 9$
24. $x^2 - y^2 = 9$
25. $x^2 + y^2 = a^2$
26. $x^2 + y^2 - 2ax = 0$
27. $y = 8$
28. $x = 12$
29. $3x - y + 2 = 0$
30. $xy = 4$
31. $y^2 = 9x$
32. $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$

Polar-to-Rectangular Conversion **In Exercises 33–42, convert the polar equation to rectangular form and sketch its graph.**

33. $r = 4$
34. $r = -5$
35. $r = 3 \sin \theta$
36. $r = 5 \cos \theta$
37. $r = \theta$
38. $\theta = \frac{5\pi}{6}$
39. $r = 3 \sec \theta$
40. $r = 2 \csc \theta$
41. $r = \sec \theta \tan \theta$
42. $r = \cot \theta \csc \theta$

Graphing a Polar Equation **In Exercises 43–52, use a graphing utility to graph the polar equation. Find an interval for θ over which the graph is traced *only once*.**

43. $r = 2 - 5 \cos \theta$
44. $r = 3(1 - 4 \cos \theta)$
45. $r = 2 + \sin \theta$
46. $r = 4 + 3 \cos \theta$
47. $r = \frac{2}{1 + \cos \theta}$
48. $r = \frac{2}{4 - 3 \sin \theta}$
49. $r = 2 \cos\left(\frac{3\theta}{2}\right)$
50. $r = 3 \sin\left(\frac{5\theta}{2}\right)$
51. $r^2 = 4 \sin 2\theta$
52. $r^2 = \frac{1}{\theta}$

53. **Verifying a Polar Equation** Convert the equation

    $$r = 2(h \cos \theta + k \sin \theta)$$

    to rectangular form and verify that it is the equation of a circle. Find the radius and the rectangular coordinates of the center of the circle.

54. **Distance Formula**
    (a) Verify that the Distance Formula for the distance between the two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

    $$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}.$$

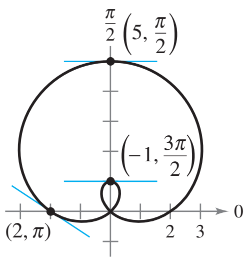
    (b) Describe the positions of the points relative to each other for $\theta_1 = \theta_2$. Simplify the Distance Formula for this case. Is the simplification what you expected? Explain.
    (c) Simplify the Distance Formula for $\theta_1 - \theta_2 = 90°$. Is the simplification what you expected? Explain.
    (d) Choose two points on the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the Distance Formula again. Discuss the result.

Distance Formula **In Exercises 55–58, use the result of Exercise 54 to approximate the distance between the two points in polar coordinates.**

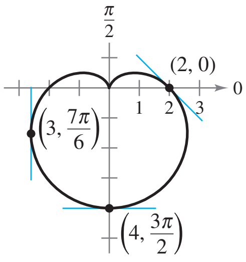
55. $\left(1, \frac{5\pi}{6}\right)$, $\left(4, \frac{\pi}{3}\right)$
56. $\left(8, \frac{7\pi}{4}\right)$, $(5, \pi)$
57. $(2, 0.5)$, $(7, 1.2)$
58. $(4, 2.5)$, $(12, 1)$

Finding Slopes of Tangent Lines In Exercises 59 and 60, find dy/dx and the slopes of the tangent lines shown on the graph of the polar equation.

59. $r = 2 + 3 \sin \theta$



60. $r = 2(1 - \sin \theta)$



Finding Slopes of Tangent Lines In Exercises 61–64, use a graphing utility to (a) graph the polar equation, (b) draw the tangent line at the given value of θ, and (c) find dy/dx at the given value of θ. (*Hint:* Let the increment between the values of θ equal $\pi/24$.)

61. $r = 3(1 - \cos \theta),\ \theta = \dfrac{\pi}{2}$

62. $r = 3 - 2 \cos \theta,\ \theta = 0$

63. $r = 3 \sin \theta,\ \theta = \dfrac{\pi}{3}$

64. $r = 4,\ \theta = \dfrac{\pi}{4}$

Horizontal and Vertical Tangency In Exercises 65 and 66, find the points of horizontal and vertical tangency (if any) to the polar curve.

65. $r = 1 - \sin \theta$

66. $r = a \sin \theta$

Horizontal Tangency In Exercises 67 and 68, find the points of horizontal tangency (if any) to the polar curve.

67. $r = 2 \csc \theta + 3$

68. $r = a \sin \theta \cos^2 \theta$

Tangent Lines at the Pole In Exercises 69–76, sketch a graph of the polar equation and find the tangents at the pole.

69. $r = 5 \sin \theta$

70. $r = 5 \cos \theta$

71. $r = 2(1 - \sin \theta)$

72. $r = 3(1 - \cos \theta)$

73. $r = 4 \cos 3\theta$

74. $r = -\sin 5\theta$

75. $r = 3 \sin 2\theta$

76. $r = 3 \cos 2\theta$

Sketching a Polar Graph In Exercises 77–88, sketch a graph of the polar equation.

77. $r = 8$

78. $r = 1$

79. $r = 4(1 + \cos \theta)$

80. $r = 1 + \sin \theta$

81. $r = 3 - 2 \cos \theta$

82. $r = 5 - 4 \sin \theta$

83. $r = 3 \csc \theta$

84. $r = \dfrac{6}{2 \sin \theta - 3 \cos \theta}$

85. $r = 2\theta$

86. $r = \dfrac{1}{\theta}$

87. $r^2 = 4 \cos 2\theta$

88. $r^2 = 4 \sin \theta$

Asymptote In Exercises 89–92, use a graphing utility to graph the equation and show that the given line is an asymptote of the graph.

	Name of Graph	Polar Equation	Asymptote
89.	Conchoid	$r = 2 - \sec \theta$	$x = -1$
90.	Conchoid	$r = 2 + \csc \theta$	$y = 1$
91.	Hyperbolic spiral	$r = 2/\theta$	$y = 2$
92.	Strophoid	$r = 2 \cos 2\theta \sec \theta$	$x = -2$

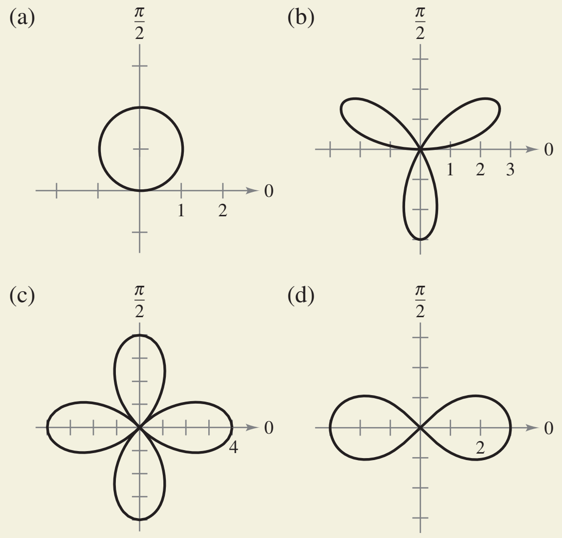
WRITING ABOUT CONCEPTS

93. Comparing Coordinate Systems Describe the differences between the rectangular coordinate system and the polar coordinate system.

94. Coordinate Conversion Give the equations for the coordinate conversion from rectangular to polar coordinates and vice versa.

95. Tangent Lines How are the slopes of tangent lines determined in polar coordinates? What are tangent lines at the pole and how are they determined?

96. HOW DO YOU SEE IT? Identify each special polar graph and write its equation.



97. Sketching a Graph Sketch the graph of $r = 4 \sin \theta$ over each interval.

(a) $0 \le \theta \le \dfrac{\pi}{2}$ (b) $\dfrac{\pi}{2} \le \theta \le \pi$ (c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

98. Think About It Use a graphing utility to graph the polar equation $r = 6[1 + \cos(\theta - \phi)]$ for (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$. Use the graphs to describe the effect of the angle ϕ. Write the equation as a function of $\sin \theta$ for part (c).

99. Rotated Curve Verify that if the curve whose polar equation is $r = f(\theta)$ is rotated about the pole through an angle ϕ, then an equation for the rotated curve is $r = f(\theta - \phi)$.

100. Rotated Curve The polar form of an equation of a curve is $r = f(\sin\theta)$. Show that the form becomes

(a) $r = f(-\cos\theta)$ if the curve is rotated counterclockwise $\pi/2$ radians about the pole.

(b) $r = f(-\sin\theta)$ if the curve is rotated counterclockwise π radians about the pole.

(c) $r = f(\cos\theta)$ if the curve is rotated counterclockwise $3\pi/2$ radians about the pole.

Rotated Curve In Exercises 101–104, use the results of Exercises 99 and 100.

101. Write an equation for the limaçon $r = 2 - \sin\theta$ after it has been rotated by the given amount. Use a graphing utility to graph the rotated limaçon for (a) $\theta = \pi/4$, (b) $\theta = \pi/2$, (c) $\theta = \pi$, and (d) $\theta = 3\pi/2$.

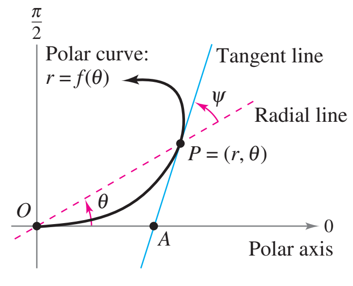
102. Write an equation for the rose curve $r = 2\sin 2\theta$ after it has been rotated by the given amount. Verify the results by using a graphing utility to graph the rotated rose curve for (a) $\theta = \pi/6$, (b) $\theta = \pi/2$, (c) $\theta = 2\pi/3$, and (d) $\theta = \pi$.

103. Sketch the graph of each equation.

(a) $r = 1 - \sin\theta$ (b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

104. Prove that the tangent of the angle ψ $(0 \le \psi \le \pi/2)$ between the radial line and the tangent line at the point (r, θ) on the graph of $r = f(\theta)$ (see figure) is given by

$$\tan\psi = \left|\frac{r}{dr/d\theta}\right|.$$



Finding an Angle In Exercises 105–110, use the result of Exercise 104 to find the angle ψ between the radial and tangent lines to the graph for the indicated value of θ. Use a graphing utility to graph the polar equation, the radial line, and the tangent line for the indicated value of θ. Identify the angle ψ.

	Polar Equation	**Value of θ**
105.	$r = 2(1 - \cos\theta)$	$\theta = \pi$
106.	$r = 3(1 - \cos\theta)$	$\theta = \dfrac{3\pi}{4}$
107.	$r = 2\cos 3\theta$	$\theta = \dfrac{\pi}{4}$
108.	$r = 4\sin 2\theta$	$\theta = \dfrac{\pi}{6}$
109.	$r = \dfrac{6}{1 - \cos\theta}$	$\theta = \dfrac{2\pi}{3}$
110.	$r = 5$	$\theta = \dfrac{\pi}{6}$

True or False? In Exercises 111–114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

111. If (r_1, θ_1) and (r_2, θ_2) represent the same point on the polar coordinate system, then $|r_1| = |r_2|$.

112. If (r, θ_1) and (r, θ_2) represent the same point on the polar coordinate system, then $\theta_1 = \theta_2 + 2\pi n$ for some integer n.

113. If $x > 0$, then the point (x, y) on the rectangular coordinate system can be represented by (r, θ) on the polar coordinate system, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$.

114. The polar equations $r = \sin 2\theta$, $r = -\sin 2\theta$, and $r = \sin(-2\theta)$ all have the same graph.

SECTION PROJECT

Anamorphic Art

Anamorphic art appears distorted, but when the art is viewed from a particular point or is viewed with a device such as a mirror, it appears to be normal. Use the anamorphic transformations

$$r = y + 16 \quad \text{and} \quad \theta = -\frac{\pi}{8}x, \quad -\frac{3\pi}{4} \le \theta \le \frac{3\pi}{4}$$

to sketch the transformed polar image of the rectangular graph. When the reflection (in a cylindrical mirror centered at the pole) of each polar image is viewed from the polar axis, the viewer will see the original rectangular image.

(a) $y = 3$ (b) $x = 2$ (c) $y = x + 5$ (d) $x^2 + (y - 5)^2 = 5^2$

This example of anamorphic art is from the Millington-Barnard Collection at the University of Mississippi. When the reflection of the transformed "polar painting" is viewed in the mirror, the viewer sees the distorted art in its proper proportions.

FOR FURTHER INFORMATION For more information on anamorphic art, see the article "Anamorphisms" by Philip Hickin in the *Mathematical Gazette*.

From the Millington & Barnard Collection of Scientific Apparatus, ca. 1855 The University of Mississippi Museum, Oxford, Mississippi